Theorems for the Anisotropic Heisenberg Ferromagnet

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The Lee-Yang theorem[1)] has been expected to hold for the Heisenberg ferromagnet[2)] and proved for special cases.[(2)]

Recently, the theorem was proved generally.[3)] Here, the main part of the proof is summarized.

We consider the anisotropic Heisenberg model with the Hamiltonian:

$$H = - \sum_{n \geq i \geq j \geq 1} J_{ij} H_{ij},$$

where

$$H_{ij} = \frac{1}{2}(\sigma_i^z \sigma_j^z - 1) + \frac{1}{2}\gamma_{ij}(\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y),$$

$$J_{ij} = J_{ji} > 0, \quad 1 > \gamma_{ji} = \gamma_{ij} > -1$$

and σ's denote Pauli matrices.

Our aim is to prove that the zeros of the partition function Q(Z) defined by

$$Q(z) = [\mathrm{Tr} \ \exp(-H/T) \ z^M]$$

where

$$M = \sigma_1^z + .. + \sigma_n^z, \text{ and } z = \exp(h/T).$$

h being the magnetic field, are all located on the unit circle in the complex z-plane.

The clue in the proof is to reduce the problem to the 2-body problem due to the general properties of the function satisfying the Lee-Yang lemma[1)] and with the aid of the Trotter formula:[4)]

$$\exp\left(\sum_{i=1}^{m} A_i\right) = \lim_{N\to\infty} \left[\prod_{i=1}^{m} \exp(A_i/N)\right]^N,$$

where all the A_i are finite matrices.

Consider the finite perturbation series of $Q(z)$ due to the Trotter formula,

$$Q_N(z) = \sum_{s_i=\pm 1} \langle \{s\}|P^N|\{s\}\rangle \, z^{s_1+\cdots+s_n},$$

where

$$P = \exp(J_{n\,n-1}H_{n\,n-1}/TN)\cdots\exp(J_{12}H_{12}/TN),$$

and $\{s\}$ denotes the set of eigenvalues of σ_i^z, $(s_1,\ldots,s_n)$. Then we shall prove the Lee-Yang theorem for $Q_N(z)$ with the all positive integer N.

Since $z^nQ_N(z)$ is a polynomial in z of the order 2n and all the coefficients tend to those of $z^nQ(z)$, which is also a polynomial of the same order, (the coefficients of z^{2n} are both equal to one) as N tends to infinity, continuity of the roots of polynomials with respect to coefficients* follows the Lee-Yang theorem for $Q(z)$.

According to the inversion symmetry in the spin space, it is sufficient to prove that the function $G_N(\{z\})$ $(\{z\}) = z_1,\ldots,z_n)$ defined by

$$G_N(\{z\}) = \Sigma\langle\{s\}|P^N|\{s\}\rangle \prod_{i=1}^{n} z_i^{s_i},$$

satisfies the Lee-Yang lemma.[1)]

The obstacle to generalize the methods in the Ising model, the Hamiltonian conserving each spin no longer, can be overcome by considering the general properties of the functions satisfying the Lee-Yang lemma.

* M. Fujiwara, Algebra vol 1 (Uchidarokakuho, Tokyo 1966) p.347
T. Takagi, The Lecture on Algebra (Kyoritsu, Tokyo, 1964) p.56
The theorem states that all the roots of the polynomial; $a_0z^n + \cdots + a_n = 0$, $(a_0 \neq 0)$, are continuous functions of $\{a_i\}$. The theorem is the Hurwitz theorem for the case of polynomials in the latter.

Definition1. The function f of $\{z\}$ is denoted by $f \in L^s(\{z\})$, Provided f satisfies the following:

(I) The function f can be written as

$$f(\{z\}) = \sum_{s_i=\pm 1} a(\{s\}) \prod_{i=1}^{n} z_i^{s_i},$$

where

$a(\{1\}) \neq 0$,

(II) $f(\{z\}) \neq 0$ as $|Z_i|$ are all ≥ 1.

Definition 2. The function f of $\{z\}$ is denoted by $f \in L(\{z\})$, provided f satisfies (I) of (D.1) and

(III) $f(\{z\}) \neq 0$ as $|Z_i|$ are all ≥ 1 and $|z_k| > 1$.

Definition 3. The operators D and d are defined as follows:

$$D(z_i, z_j) f(\{z\}) = \Sigma a(\ s\ ) \prod_{k \neq j} z_k^{s_k} \delta(s_i, s_j),$$

and

$$d_i = z_i \frac{\partial}{\partial z_i}$$

Then the following theorems can be proved.

Theorem 1. The function f of $\{z\}$, satisfying (I) of (D.1), belongs to $L(\{z\})$ (res. L^s) if and only if f can be written as a function of every z_i in the following form:

$$f(\{z\}) = A_i(z_i - a_i/z_i),$$

where

$A_i \in L^s(\{z_j\}_{j \neq i})$ and $|a_i| \leq 1$ (res. < 1 for L^s) as $|z_j|(j \neq i)$ are all ≥ 1.

Theorem 2. The coefficient of $z_1 z_2 .. z_m$ of $f \in L(\{z\})$ belongs to $L^s(\{z_j\}_{j \neq 1,2,..,m})$.

Theorem 3. The function f of $\{z\}$, satisfying (I) of (D.1), belongs to $L(\{z\})$ if and only if f can be written as a function of z_1 in the following form:

$$f(\{z\}) = A(z_1 - a/z_1),$$

where
$A \in L^s(\{z_j\}_{j \neq 1})$, $|a| \leq 1$ as $|z_j|(j \neq 1)$ are all ≥ 1 and $|a| < 1$ as $|z_j|(j \neq 1)$ are all ≥ 1 and $|z_k| > 1 (k \neq 1)$.

From these theorems, it results that operators D and d map L into L.

Theorem 4. Provided f and g belong to $L(\{z\})$ and $L(\{y\})$ respectively, the functions $D(z_i, z_j)f$, $D(z_i, y_j)$ (fg) and $d_i f$ belong to $L(\{z_k\}_{k \neq j})$, $L(\{z\}, \{y_k\}_{k \neq j})$ and $L(\{z\})$ respectively.

In fact, let f and g be in the form as in (T. 3), in which $\{y\}$, B and b replace $\{z\}$, A and a for g. Then we have

$$D(z_1, y_1)\ (fg) = AB(z_1 + ab/z_1),$$

and it is easily shown from (T. 3) that Ab and ab satisfy the demands in (T. 3). The remaining part of the theorem can be shown in the similar way.

By direct calculation, the function with the 2-spin system defined by

$$F(z_1, z_2; y_1, y_2: K_{12}) = \Sigma\ z_1^{s_1} z_2^{s_2} < s_1 s_2 | \exp(K_{12} H_{12}) | s_1' s_2' >$$

$$\times\ y_1^{s_1'} y_2^{s_2'},$$

can be proved to belong to $L(z_1, z_2, y_1, y_2)$ as K_{12} is positive.

With the aid of (T. 4), we obtain the 3-spin function satisfying the Lee-Yang lemma by coupling the two 2-spin functions with the first and second and the first and third spins respectively.

In fact, we have from (T. 4)

$$D(u_1, v_1)\ [F(z_1, z_2; u_1, y_2: K_{12}) F(v_1, z_3; y_1, y_3: K_{13})]$$

$$\in L(\{z\}, u_1, \{y\}).$$

It is apparent from (D. 1), (D. 2) and (T. 2) that $f(\{z\}; \{1\})$ belongs to $L(\{z\})$ if $f(\{z\}; \{u\})$ belongs to $L(\{z\}, \{u\})$. By taking u_1 to be equal to one, we obtain

$$F(z_1, z_2, z_3; y_1, y_2, y_3: K_{12}K_{13}) = \sum z_1^{s_1} z_2^{s_2} z_3^{s_3} < \{s\} | \exp(K_{12}H_{12})$$

$$\times \exp(K_{13}H_{13}) | \{s'\} > y_1^{s'_1} y_2^{s'_2} y_3^{s'_3} \in L(\{z\}, \{y\}).$$

Repeating the similar argument, we can successively connect the 2-spin functions by using the operator D as a coupler of the train, and obtain

Theorem 5. The function defined by

$$F_N(\{z\}; \{y\}) = \sum < \{s\} | P'^N | \{s'\} > \prod_{i=1}^{n} z_i^{s_i} y_i^{s'_i},$$

belongs to $L(\{z\}, \{y\})$,

where

$$P' = \exp(K_{n\ n-1}H_{n\ n-1}) \ldots \exp(K_{12}H_{12}) \text{ and } K_{ij} > 0.$$

Applying (T. 4) to F_N in (T. 5), and taking K_{ij} be equal to J_{ij}/TN, we obtain

Theorem 6. The function $G_N(\{z\})$ defined in the preceding paragraph belongs to $L(\{z\})$.

In fact, the function $G_N(\{z\})$ is equal to

$$\prod_{i=1}^{n} D(z_i, y_i) F_N(\{z\}; \{y\}).$$

Hence the proof of the Lee-Yang theorem is completed.

The Lee-Yang lemma is by no means merely a tool for obtaining the Lee-Yang theorem.[5] The first Griffiths inequality is derived from the Lee-Yang lemma.[3)6)]

Theorem 7. The absolute magnitude of $f \in L(\{z\})$ is an increasing function of every $|z_i|$ as $|z_j|$ are all ≥ 1.

Theorem 8.[3)6)] Provided a function f belongs to $L(\{z\})$ and real and ≥ 0 as z_i are all ≥ 1, $d_1 f \geq 0$, $d_1 d_2 f \geq 0, \ldots, d_1 \ldots d_n f \geq 0$ as z_i are all ≥ 1.

All the coefficients $\langle\{s\}|P^N|\{s\}\rangle$ of $G_N(\{z\})$ tend to $\langle\{s\}|\exp(-H/T)|\{s\}\rangle$, which is positive, as N tends to infinity and there are only finite number of values of $\{s\}$. Then $\langle\{s\}|P^N|\{s\}\rangle$ are all ≥ 0 for sufficiently large N. From (T. 6) and (T. 8), we obtain the first Griffiths inequality:

$$\langle \sigma_1^z \sigma_2^z \ldots \sigma_m^z \rangle \geq 0 \quad (m=1, 2, \ldots, n).$$

Due to continuity, the main results, the Lee-Yang theorem and the Griffiths inequality still hold when some of J's vanish or γ's are equal to ± 1.

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